

# On the evolution equations for a self-gravitating charged scalar field

Daniela Pugliese · Juan A. Valiente  
Kroon

Received: date / Accepted: date

**Abstract** We consider a complex scalar field minimally coupled to gravity and to a  $U(1)$  gauge symmetry and we construct of a first order symmetric hyperbolic evolution system for the Einstein-Maxwell-Klein-Gordon system. Our analysis is based on a 1+3 tetrad formalism which makes use of the components of the Weyl tensor as one of the unknowns. In order to ensure the symmetric hyperbolicity of the evolution equations, implied by the Bianchi identity, we introduce a tensor of rank 3 corresponding to the covariant derivative of the Faraday tensor, and two tensors of rank 2 for the covariant derivative of the vector potential and the scalar field.

**Keywords** Scalar field; initial value problem

## 1 Introduction

In this article we discuss the construction of suitable evolution equations for a self-gravitating charged scalar field governed by the so-called *Einstein-Maxwell-Klein-Gordon system*. It is well known that General Relativity admits initial value problem formulation whereby one prescribes certain initial data on a 3-dimensional hypersurface, and one purports to reconstruct the spacetime associated to this initial data —a so-called Cauchy problem. The formulation of an initial value problem is a natural starting point for a wide variety of analytical studies of the qualitative properties of the solutions to the equations

---

Daniela Pugliese  
School of Mathematical Sciences, Queen Mary University of London  
Mile End Road, London E1 4NS, UK  
E-mail: dpugliese@maths.qmul.ac.uk

Juan A. Valiente Kroon  
School of Mathematical Sciences, Queen Mary University of London  
Mile End Road, London E1 4NS, UK  
E-mail: j.a.valiente-kroon@qmul.ac.uk

and a necessary starting point for the construction of the numerical solutions. Examples of qualitative aspects of the solutions requiring an suitable initial value formulation are the discussion of local and global existence problems and the analysis of the stability of certain reference solutions.

In this article we approach the construction of the evolution equations of the Einstein-Maxwell-Klein-Gordon system from the point of view of mathematical Relativity. Hence, the amenability of our analysis to analytic considerations takes precedence over numerical considerations. Our discussion is chiefly concerned with the construction of a first order system of quasilinear hyperbolic evolution equations (*FOSH*) out of the Einstein-Maxwell-Klein-Gordon equations. In general, *FOSH* systems can be written in the following form:

$$\mathbf{A}^0 \partial_t \mathbf{v} - \mathbf{A}^j \partial_j \mathbf{v} = \mathbf{B} \mathbf{v}, \quad (1)$$

where the variables of the system are collected in a  $n$ -dimensional vector  $\mathbf{v}$ ,  $t$  denotes a suitable time coordinate and  $\mathbf{x} = (x^1, x^2, x^3)$  denotes some spatial coordinates and  $j = 1, 2, 3$ . The matrices  $\mathbf{A}^0$  and  $\mathbf{A}^j$  are matrix valued functions depending on the coordinates  $(t, \mathbf{x})$  and the unknown  $\mathbf{v}$ —that is, the system (1) is, in general, *quasilinear*. The system is said to be *symmetric hyperbolic* if the matrices  $\mathbf{A}^0$  and  $\mathbf{A}^j$  are symmetric and if  $\mathbf{A}^0$  is a negative-definite matrix.

As a consequence of the construction presented in this article, one automatically obtains a local existence and uniqueness result for the equations of the Einstein-Maxwell-Klein-Gordon system. The symmetric hyperbolicity of the resulting evolution equations ensures that the Cauchy problem for the systems is well-posed. In other words, if one prescribes suitable initial data on an initial hypersurface, a unique solution exists in a neighborhood of that hypersurface—local in time existence. The solutions depend continuously on the values of initial data [1]. The question of the well-posedness of the evolution equations of the Einstein-Maxwell-Klein-Gordon system is a problem that touches upon many aspects of current theoretical and numerical analysis of many physical phenomena. An suitable point of entry for the extensive literature in this topic, with particular emphasis on numerics, is given in [2]. Scalar fields enter as important ingredients in many theoretical models of contemporary physics. Here, we consider a self gravitating scalar field, minimally coupled with the gravitational field via the Einstein equations and the electromagnetic field by the coupling constant  $q$ —the so-called *scalar boson charge*. In the presence of strong gravitational fields, scalar fields are described by the general-relativistic field equations. These configurations arise in many areas of high-energy Cosmology as inflationary models carried by scalar fields. Moreover, scalar fields are also candidates for dark matter source. In astrophysics, scalar fields arise in stellar models as configurations of hypothetical scalar particles known as bosons stars, or as a boson core of very compact stars [3].

Our analysis is based on a 1+3 tetrad formalism. The hyperbolic reduction procedure described in the present article borrows from the discussion of the

evolution equations for the Einstein-Euler system by H. Friedrich in [4] —see also [5, 6, 7] and the generalization to Einstein-Euler-Maxwell system in [8]. In this reference a Lagrangian gauge was used to construct the required hyperbolic reduction and to obtain the desired evolution equations. The central equation in this discussion is the *Bianchi identity*. It provides evolution equations for the components of the Weyl tensor. The addition of electromagnetic interactions to Friedrich's system in [8] destroys, in principle, the symmetric hyperbolic nature of the evolution equation as derivatives of the Faraday tensor enter into the principal part of the Bianchi evolution equations. This difficult was handled by the introduction of a new field unknown corresponding to the derivative of the Faraday tensor for which suitable field and evolution equations can be obtained. This strategy can be adapted to the case of a complex scalar field case coupled with the electromagnetic field. In addition to the auxiliary variable associated to the covariant derivative of the Faraday tensor, our analysis required the use of two further tensors of rank 2 for the covariant derivative of the vector potential and the scalar field. The Lagrangian approach to the description of a scalar field follows the ideas developed in [9] for a real scalar field. It is important to observe that the extra gauge freedom induced by the frame representation of the Einstein-Klein-Gordon system used in the present article is associated to the evolution of the spatial frame coefficients along the flow of the time-like frame —in particular, in order to implement the Lagrangian description it is required that the timelike vector of the orthonormal frame follows the matter flow lines. In order to do this the scalar field must satisfy suitable regularity conditions. In fact, the local nature of the treatment (fixing suitable initial data existence and uniqueness of a solution to the evolution equation can be established locally in time) assures the scalar gradient to be timelike only in a neighborhood of the initial data. Afterwards this conditions may no longer be satisfied and the the scalar gradient can be spacelike or null, and in this case the entire set up breaks down. The remaining frame components are chosen to be Fermi propagated along this direction.

The present article is structured as follows: In Section 2 we write and discuss the relativistic equations describing a charged scalar field. The tetrad formalism used in this article is briefly reviewed in Section 3. General remarks concerning the reduction procedure to obtain suitable evolution equations are given in Section 4; the resulting evolution equations are discussed in Section 4.2 and the subsequent sections. The auxiliary fields and the gauge conditions are presented in Section 4.6. A summary of the evolution equations is given in Section 4.12. Some concluding remarks are given in Section 5.

## 2 The charged scalar field equations

In the present article we will consider the *Einstein field equations*

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2)$$

with matter source given by a self-gravitating charged (i.e. complex) scalar field  $\Phi$  minimally coupled to gravity and to a  $U(1)$  gauge field  $A^\mu$ . The total energy momentum tensor for this system is given by

$$T_{\mu\nu} = T_{\mu\nu}^{(\text{KG})} + T_{\mu\nu}^{(\text{em})}, \quad (3)$$

where, using units such that  $\hbar = c = 1$ ,

$$T_{\mu\nu}^{(\text{em})} \equiv \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} - g^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho}, \quad (4)$$

denotes the energy momentum tensor of the *free electromagnetic field* and

$$T_{\mu\nu}^{(\text{KG})} \equiv (D_\mu \Phi)^* (D_\nu \Phi) + (D_\nu \Phi) (D_\mu \Phi)^* - g_{\mu\nu} g^{\lambda\rho} (D_\lambda \Phi)^* (D_\rho \Phi) \quad (5)$$

is the energy momentum tensor for the charged scalar field —see e.g. [10]. In the previous equations  $D_\mu \equiv \nabla_\mu + iqA_\mu$ , where the constant  $q$  is the *boson charge* and  $\nabla_\mu$  stands for the Levi-Civita covariant derivative of the metric  $g_{\mu\nu}$  of signature  $(+, -, -, -)$ . Moreover,  $*$  denotes the operation of complex conjugation while

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad (6)$$

is the electromagnetic field tensor (*Faraday tensor*).

In the sequel, it will be convenient to write the scalar field  $\Phi$  in terms of two real scalar fields  $\theta$  and  $\phi$  such that

$$\Phi(x) = e^{i\theta(x)} \phi(x), \quad \phi^* = \phi, \quad \theta^* = \theta. \quad (7)$$

The energy momentum tensor of equation (5) then reads

$$T_{\mu\nu}^{(\text{KG})} = \frac{\phi^2}{2} \left( \frac{2}{\phi^2} \nabla_\mu \phi \nabla_\nu \phi + 2 (\sigma_\mu \sigma_\nu + q^2 A_\mu A_\nu + 2q\sigma_{(\mu} A_{\nu)}) \right) \quad (8)$$

$$- g_{\mu\nu} \left( \frac{1}{\phi^2} g^{\lambda\rho} \nabla_\lambda \phi \nabla_\rho \phi + \sigma^2 + q^2 A^2 + 2q\sigma_\lambda A^\lambda \right), \quad (9)$$

where the vector field  $\sigma_\mu$  is defined by  $\sigma_\mu \equiv \nabla_\mu \theta$ , and we have used the notation  $A^2 \equiv A_\lambda A^\lambda$ . Consequently, it is possible to write the tensor  $T_{\mu\nu}^{(\text{KG})}$  as

$$T_{\mu\nu}^{(\text{KG})} = T_{\mu\nu}^{(\phi)} + \frac{1}{2} \phi^2 T_{\mu\nu}^{(\sigma)} + \frac{1}{2} \phi^2 T_{\mu\nu}^{(\text{A})} + \frac{1}{2} \phi^2 T_{\mu\nu}^{(\sigma\text{A})}, \quad (10)$$

where

$$T_{\mu\nu}^{(\phi)} \equiv \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\lambda\rho} \nabla_\lambda \phi \nabla_\rho \phi, \quad (11)$$

$$T_{\mu\nu}^{(\sigma)} \equiv 2\sigma_\mu \sigma_\nu - g_{\mu\nu} \sigma^2, \quad (12)$$

$$T_{\mu\nu}^{(\text{A})} \equiv q^2 (2A_\mu A_\nu - g_{\mu\nu} A^2), \quad (13)$$

$$T_{\mu\nu}^{(\sigma\text{A})} \equiv 2q (2\sigma_{(\mu} A_{\nu)} - g_{\mu\nu} \sigma_\lambda A^\lambda). \quad (14)$$

Consistent with the requirement that  $\nabla^\mu T_{\mu\nu} = 0$ , we require the scalar field  $\Phi$  to satisfy the *Klein-Gordon equation*

$$D_\mu D^\mu \Phi = 0.$$

Equivalently, one has that

$$\nabla_\mu \nabla^\mu \Phi + 2iA^\mu \nabla_\mu \Phi + ig^{\mu\nu} \nabla_\mu A_\nu \Phi - A^2 \Phi = 0. \quad (15)$$

In what follows, for simplicity of the presentation, we set  $q = 1$ .

Written in terms of the real fields  $\phi$  and  $\sigma_\mu$ , equation (15) reads

$$\frac{1}{\phi} \nabla^\mu \nabla_\mu \phi - \sigma_\mu \sigma^\mu - A^2 - 2A^\mu \sigma_\mu = 0, \quad (16)$$

$$\nabla^\mu \sigma_\mu + \frac{1}{\phi} \sigma_\mu \nabla^\mu \phi + \frac{1}{\phi} \sigma_\mu \nabla^\mu \phi + \frac{2}{\phi} A^\mu \nabla_\mu \phi + g^{\mu\nu} (\nabla_\mu A_\nu) = 0. \quad (17)$$

Finally, the electromagnetic field is described by the *Maxwell equations* in the form

$$\nabla_{[\mu} F_{\nu\lambda]} = 0, \quad \nabla^\nu F_{\mu\nu} = \phi^2 (\sigma_\mu + A_\mu). \quad (18)$$

### 3 Tetrad formalism

In the present article, the Einstein fields equations will be expressed in terms of a frame formalism introduced in [4]. To this end, let  $\{e_a\}_{a=0,\dots,3}$  denote a basis of *frame vectors* on the spacetime  $(\mathcal{M}, g_{\mu\nu})$  satisfying  $g_{ab} \equiv g(e_a, e_b) = \eta_{ab} = \text{diag}(1, -1, -1, -1)$ . We denote by  $(\omega^a)$  the corresponding *dual basis* (cobasis). Here, and in the rest of the article, Latin letters  $a, b, \dots$  are used as spacetime frame indices taking the values  $0, \dots, 3$  while Greek letters  $\mu, \nu, \dots$  denote the tensorial character of each object —i.e. they are spacetime indices. The Latin letters  $i, j, k, \dots$  will be used as spatial frame indices taking the values  $1, 2, 3$ .

The frame fields  $e_a$  and the cobasis  $\omega^a$  are expressed in terms of a local coordinate basis as

$$e_a = e_a^\mu \partial_\mu, \quad \omega^a = \omega^a_\mu dx^\mu. \quad (19)$$

Thus,

$$\omega^a_\mu e_b^\mu = \delta_b^a, \quad \omega^a_\mu e_a^\nu = \delta_\mu^\nu, \quad (20)$$

so that the metric tensor can be written as

$$g_{\mu\nu} = \eta_{ab} \omega^a_\mu \omega^b_\nu, \quad g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu. \quad (21)$$

The *commutation coefficients*  $D_{ab}^c$  for the tetrad  $e_a^\alpha$  are defined by

$$[e_a, e_b] = D_{ab}^c e_c \equiv (e_a^\mu \partial_\mu e_b^\nu - e_b^\mu \partial_\mu e_a^\nu) e_\nu. \quad (22)$$

Finally, the *connection coefficients* (or *Ricci rotation coefficients*)  $\Gamma_b^a{}_c$  for the tetrad  $e_a^\alpha$  are defined by the relations

$$\nabla_a e_b = \Gamma_a^c{}_b e_c, \quad \nabla_a \omega^b = -\Gamma_a^b{}_c \omega^c. \quad (23)$$

Hence, equation (22) can be rewritten as

$$[e_a, e_b] = (\Gamma_a^c{}_b - \Gamma_b^c{}_a) e_c. \quad (24)$$

In particular, since  $e_a(\eta_{bc}) = 0$ , one has that

$$\Gamma_{a(bc)} = 0. \quad (25)$$

The components of the *Riemann tensor* with respect to the frame  $e_a$  are given in terms of the connection coefficients by

$$R_{bcd}^a = e_c(\Gamma_d^a{}_b) - e_d(\Gamma_c^a{}_b) - \Gamma_e^a{}_b(\Gamma_c^e{}_d - \Gamma_d^e{}_c) + \Gamma_c^a{}_e \Gamma_d^e{}_b - \Gamma_d^a{}_e \Gamma_c^e{}_b. \quad (26)$$

The components of the Riemann tensor admit the decomposition

$$R_{abcd} = C_{abcd} + \{g_{a[c} S_{d]b} - g_{b[c} S_{d]a}\}, \quad (27)$$

where  $S_{ab}$  denotes the *Schouten tensor*

$$S_{ab} \equiv R_{ab} - \frac{1}{6} R g_{ab}, \quad (28)$$

with  $R_{ab} \equiv R^c{}_{acb}$  the components of the *Ricci tensor* and  $R \equiv g^{ab} R_{ab}$  the *Ricci scalar*. Finally  $C_{abcd}$  denotes the components of the *Weyl tensor* with respect to  $e_a$ . The components of the curvature tensor satisfy the Bianchi identity

$$\nabla_{[e} R_{bc]d}^a = 0. \quad (29)$$

As it is well known, the contracted version of the above identity leads to

$$\nabla^a G_{ab} = 0, \quad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab}, \quad (30)$$

where  $G_{ab}$  denotes the components of the *Einstein tensor*. Defining the *Friedrich tensor*  $F_{abcd}$  via

$$F_{abcd} \equiv C_{abcd} - g_{a[c} S_{d]b}, \quad (31)$$

one concludes, furthermore, that

$$F_{abc} \equiv \nabla_d F_{abc}^d = 0. \quad (32)$$

Taking the Hodge dual of equation (32) with respect to the index pair  $cd$ , we obtain another equation of the form (32) for a tensor  $\tilde{F}_{abcd}$  defined by

$$\tilde{F}_{abcd} \equiv C_{abcd}^* + \frac{1}{2} S_{pb} \epsilon^p{}_{acd}, \quad (33)$$

where  $\epsilon_{abcd}$  denotes the components of the *completely antisymmetric Levi Civita tensor* with respect to the frame  $e_a$  and  $C_{abcd}^* \equiv \frac{1}{2} C_{abef} \epsilon^{ef}{}_{cd}$  —see e.g. [4, 11, 12, 13].

## 4 The hyperbolic reduction procedure

### 4.1 General considerations regarding Friedrich's frame formulation of the Einstein field equations

Following [4], it is convenient to introduce here the following notation:

$$N^a \equiv \delta_0^a, \quad N \equiv N^a e_a = e_0. \quad (34)$$

A tensor  $T_{a_1 \dots a_p}$  is said to be *spatial* if any contraction with  $N^a$  vanishes. The subspaces orthogonal to  $N^a$  inherit the metric  $h_{ab} \equiv g_{ab} - N_a N_b$  (indices are raised and lowered using  $g_{ab}$ ). Thus,  $h_a^b$  is the orthogonal projector into these subspaces.

For a given tensor, any contraction with  $N$  will be denoted by replacing the corresponding frame index by  $N$ . The projection of a tensor with respect to  $h_a^b$  will be indicated by  $'$ . Thus, a tensor  $T_{abc}$  one has that

$$T'_{aNb} = h_a^m N^p h_b^q T_{mpq}. \quad (35)$$

A spatial vector satisfies  $T_{a_1 \dots a_p} = T'_{a_1 \dots a_p}$ . In what follows, let  $\epsilon_{abc} \equiv \epsilon'_{Nabc}$ , where  $\epsilon_{0123} = 1$ . In terms of the latter one has the decomposition

$$\epsilon_{abcd} = 2 (N_{[a} \epsilon_{b]cd} - \epsilon_{ab[c} N_{d]}) . \quad (36)$$

Given a spatial tensor we define the spatial covariant derivative

$$\mathcal{D}_a T_{a_1 \dots a_p} \equiv \nabla_b T_{b_1 \dots b_p} h_a^{b_1} h_{a_1}^{b_2} \dots h_{a_p}^{b_p}. \quad (37)$$

In particular, it can be readily verified that  $\mathcal{D}_a h_{bc} = \mathcal{D}_a \epsilon_{bcd} = 0$ .

In the sequel, it will be convenient to introduce the tensors

$$a^a = N^b \nabla_b N^a, \quad \chi_{ab} = h_a^c \nabla_c N_b, \quad \chi \equiv h^{ab} \chi_{ab}. \quad (38)$$

These are related to the connection as can be seen from

$$\nabla_a N^b = N_a a^b + \chi_a^b, \quad a^a = h_b^a \Gamma_{00}^b, \quad \chi_{ab} = -h_a^c h_b^d \Gamma_{cd}^0. \quad (39)$$

For future use, we notice the decomposition of the Weyl tensor as

$$C_{abcd} = 2(l_{b[c} \hat{E}_{d]a} - l_{a[c} \hat{E}_{d]b}) - 2(N_{[c} \hat{B}_{d]p} \epsilon_{ab}^p + N_{[a} \hat{B}_{b]p} \epsilon_{cd}^p) \quad (40)$$

in terms of its electric  $\hat{E}_{ab} = C'_{NaNb}$  and magnetic part  $\hat{B}_{ab} = C^{*'}_{NaNb}$  relative to  $N_a$ , where  $l_{ab} \equiv h_{ab} - N_a N_b$ . The Schouten tensor  $S_{ab}$  will be expressed in terms of the matter fields. Accordingly, we write

$$S_{ab} = S_{ab}^{(\text{KG})} + S_{ab}^{(\text{em})}, \quad (41)$$

where

$$S_{ab}^{(\text{KG})} = \kappa (T_{ab}^{(\text{KG})} - \frac{1}{3} g_{ab} g^{cd} T_{cd}^{(\text{KG})}), \quad (42)$$

with a similar expression for  $S_{ab}^{(em)}$ . Using equations (40) and (41) in equation (27), the Riemann curvature tensor  $R_{abcd}$ , can be written in the form

$$R_{abcd} = R_{abcd}^W + R_{abcd}^{KG} + R_{abcd}^{em}, \quad (43)$$

where

$$R_{abcd}^{em} \equiv \kappa \left( (E^2 + B^2) \left( \frac{1}{2} (g_{a[c} g_{d]b} - g_{b[c} g_{d]a}) - 2N_{[b} g_{a][c} N_{d]} \right) - 2E_{[b} g_{a][c} E_{d]} + 2B_{[b} g_{a][c} B_{d]} + 2(N_{[a} g_{b][c} \mathcal{G}_{d]} - \mathcal{G}_{[a} g_{b][c} N_{d]} \right), \quad (44)$$

$$R_{abcd}^W \equiv 2(l_{b[c} \hat{E}_{d]a} - l_{a[c} \hat{E}_{d]b}) - 2(N_{[c} \hat{B}_{d]p} \epsilon^p_{ab} + N_{[a} \hat{B}_{b]p} \epsilon^p_{cd}), \quad (45)$$

$$R_{abcd}^{KG} \equiv g_{a[c} S_{d]b}^{KG} - g_{b[c} S_{d]a}^{KG}, \quad (46)$$

with  $S_{db}^{KG}$  as defined in equation (42). Finally, from the Bianchi identity for the tensor  $F_{abcd}$ , equation (32), we obtain the decomposition

$$F_{abc} = N_a (F'_{NbN} N_c - F'_{NcN} N_b) - 2F'_{aN[b} N_{c]} + N_a F'_{Nbc} + F'_{abc}, \quad (47)$$

where

$$F'_{aNb} = \mathcal{L}_N F'_{NaNb} + \mathcal{D}^c F'_{caNb} - a^c (F'_{Nacb} + F'_{caNb}) + a_a F'_{NNNb} - \chi^{cd} F'_{cadb} - \chi_a^c F'_{NcNb} - \chi_b^c F'_{NaNc} + \chi_a^c F'_{cNNb} + \chi F'_{NaNb} \quad (48)$$

with  $\mathcal{L}_N$  denoting the Lie derivative in the direction of  $N^a$ . From equation (31) we compute the following components for the Friedrich tensor  $F_{abcd}$  and  $\tilde{F}_{abcd}$ :

$$\tilde{F}'_{NNNa} = 0, \quad F'_{NNNa} = -\frac{\kappa}{2} T_{ec} N^c h_a^e, \quad (49)$$

$$\tilde{F}'_{NaNb} = \hat{B}_{ab}, \quad F'_{NaNb} = \hat{E}_{ab} - \frac{\kappa}{2} T_{dc} h_b^d h_a^c, \quad (50)$$

$$\tilde{F}'_{NNab} = \frac{\kappa}{2} (T_{pv} N^v) \epsilon^p_{uab} N^u, \quad F'_{NNab} = 0, \quad (51)$$

$$\tilde{F}'_{aNNb} = -\hat{B}_{ab} + \frac{\kappa}{2} (T_{pv} N^v) \epsilon^p_{aub} N^u, \quad (52)$$

$$F'_{aNNb} = -\hat{E}_{ab} + \frac{\kappa}{2} h_{ab} [T_{uv} N^v N^u], \quad (53)$$

$$\tilde{F}'_{Nabc} = \hat{E}_{ap} \epsilon^p_{bc} + \frac{\kappa}{2} (T_{pv} h_a^v) \epsilon^p_{ubc} N^u, \quad F'_{Nabc} = -\hat{B}_{ap} \epsilon^p_{bc}, \quad (54)$$

$$\tilde{F}'_{aNbc} = -\hat{E}_{ap} \epsilon^p_{bc} + \frac{\kappa}{2} (T_{up} N^u) \epsilon^p_{dvf} h_a^d h_b^v h_c^f, \quad (55)$$

$$F'_{aNbc} = \hat{B}_{ap} \epsilon^p_{bc} - \frac{\kappa}{2} [h_{ab} (T_{fu} N^u h_c^f) - h_{ac} (T_{fu} N^u h_b^f)], \quad (56)$$

$$\tilde{F}'_{abNc} = -2\hat{E}_{p[b} \epsilon_{a]}^p{}_c + \frac{\kappa}{2} T_{pu} h_b^u \epsilon^p_{avc} N^v, \quad (57)$$

$$F'_{abNc} = -\hat{B}_{cp} \epsilon^p_{ab} + \frac{\kappa}{2} h_{ac} T_{uv} N^u h_b^v, \quad (58)$$

$$\tilde{F}'_{abcd} = -\hat{B}_{pq} \epsilon^p_{ab} \epsilon^q_{cd} + \frac{\kappa}{2} (T_{pv} h_{[b}^v) \epsilon^p_{uef} h_a^u h_c^e h_d^f, \quad (59)$$

$$F'_{abcd} = 2(l_{b[c} \hat{E}_{d]a} - l_{a[c} \hat{E}_{d]b}) - \kappa T_{uv} h_b^v h_{c[a} h_{d]}^u. \quad (60)$$



## 4.2 Evolution equations for the electric and magnetic fields

The Maxwell equations are given by

$$\nabla_{[a}F_{bc]} = 0, \quad \nabla^a F_{ab} = J_b, \quad (61)$$

where

$$J_a \equiv -\phi^2(\sigma_a + A_a). \quad (62)$$

As it is well known, the electromagnetic field tensor (Faraday tensor)  $F_{ab}$  can be split into its *electric* and *magnetic* parts,  $E_a = F_{ab}N^b$  and  $B^a = \frac{1}{2}\epsilon^{abcd}N_b F_{cd}$ , components with respects to a flow. More precisely, one has that

$$F_{ab} = 2E_{[a}N_{b]} - \epsilon_{abcd}B^cN^d, \quad (63)$$

where  $N_a N^a = 1$ . Using the decomposition into electric and magnetic parts, the electromagnetic energy-momentum tensor of equation (4) can be written as

$$T_{ab}^{(\text{em})} \equiv -\frac{1}{2}N_a N_b(E^2 + B^2) + \frac{h_{ab}}{6}(E^2 + B^2) + P_{ab} - 2\mathcal{G}_{(a}N_{b)}, \quad (64)$$

where we have written  $E^2 \equiv E_a E^a$  and  $B^2 \equiv B^a B_a$ , and  $P_{ab}$  denotes the symmetric, trace-free tensor given by

$$P_{ab} \equiv \frac{h_{ab}}{3}(E^2 + B^2) - (E_a E_b + B_a B_b), \quad (65)$$

and

$$\mathcal{G}_a \equiv \epsilon_{auvd}E^u B^v N^d, \quad (66)$$

denotes the *Poynting vector*. Projecting equations (61) along direction longitudinal and transverse to the vector  $N^b$ , and orthogonal to the  $N^b$ , one obtains the *Maxwell evolution equations*

$$\dot{E}_{\langle f \rangle} = -2E^a h_{f[a} \nabla_{b]} N^b - \epsilon_{abcd} h_f{}^b \nabla^a (B^c N^d) - h_e{}^b J_b, \quad (67)$$

$$\dot{B}_{\langle f \rangle} = -2B^a h_{f[a} \nabla_{b]} N^b + \epsilon_{abcd} h_f{}^b \nabla^a (E^c N^d), \quad (68)$$

and the *Maxwell constraint equations*

$$D^a B_a = -\epsilon_{abcd} N^b E^c \nabla^a N^d, \quad (69)$$

$$D^a E_a = \epsilon_{abcd} N^a B^b \nabla^c N^d + N^b J_b, \quad (70)$$

where  $\dot{E} \equiv N^a \nabla_a E_b$  stands for the covariant time derivative of  $E_a$  along  $N$ ,  $\dot{w}_{\langle a \rangle} \equiv h_a{}^b \dot{w}_b$  is the orthogonal projection of covariant time derivatives along the  $N^a$  and  $D_a w_b \equiv h_a{}^u h_b{}^v \nabla_u w_v$  denotes the fully orthogonally projected covariant derivative of a vector  $w_a$ . Finally, we note that

$$\epsilon_{abcd} h_f{}^b \nabla^a (X^c N^d) = -\text{curl} X_f + \epsilon_{afcd} N^a X^c \dot{N}^d,$$

where  $\text{curl} X_f \equiv \epsilon_{facd} N^d \nabla^a B^c$ .

### 4.3 Evolution equation for $\hat{B}_{ab}$

The evolution equation for the magnetic part of the Weyl tensor,  $\hat{B}_{ab}$  is encoded in the component  $\tilde{F}'_{(a|N|b)}$  of the Friedrich tensor. More precisely, one has that

$$0 = \tilde{F}'_{(a|N|b)} = \mathcal{L}_N \hat{B}_{ab} - D_d \hat{E}_{c(a} \epsilon_{b)}^{dc} + 2a_c \epsilon^{cd} {}_{(a} \hat{E}_{b)d} - \chi^c {}_{(a} \hat{B}_{b)c} - 2\chi {}_{(a} {}^c \hat{B}_{b)c} + \chi \hat{B}_{ab} - \chi_{cd} \hat{B}_{pq} \epsilon^{pc} {}_{(a} \epsilon^{dq} {}_{b)} + \tilde{F}'^{(M)}_{(a|N|b)}, \quad (71)$$

where  $\tilde{F}'^{(M)}_{(a|N|b)} \equiv \tilde{F}'^{(em)}_{(a|N|b)} + \tilde{F}'^{(KG)}_{(a|N|b)}$  can be written as

$$\begin{aligned} \tilde{F}'^{(M)}_{(a|N|b)} &\equiv \frac{\kappa}{2} \mathcal{D}^c (T_{up} \epsilon^p{}_{cv} (h_a)^u N^v) - \frac{\kappa}{2} \chi^{cd} (T_{pv} h_{(a}{}^v h_{b)}^f) \epsilon^p{}_{uef} h_c{}^u h_d{}^e \\ &\quad + \frac{\kappa}{2} \epsilon^p{}_{cu} (b \chi^c{}_a) N^u T_{pv} N^v. \end{aligned} \quad (72)$$

Substituting equations (3) and (4) in equation (72) we find that the electromagnetic contribution to the evolution equation of  $\hat{B}_{ab}$  is given by

$$\begin{aligned} \tilde{F}'^{(em)}_{(a|N|b)} &\equiv -\frac{\kappa}{2} \left( \epsilon \mathcal{D}^c (N^v \epsilon^p{}_{cv} (h_a)^u F_{uq} F_p{}^q - \frac{1}{4} h_{a)p} F_{qs} F^{qs}) \right. \\ &\quad - \chi^{ue} \epsilon^p{}_{uef} (h^f{}_{(b} h_{a)}^v F_{vc} F_p{}^c - \frac{1}{4} h^f{}_{(b} h_{a)p} F_{qs} F^{qs}) \\ &\quad \left. - N^u \epsilon^p{}_{cu} (b \chi^c{}_a) E_d F_p{}^d \right). \end{aligned} \quad (73)$$

Notice that this last expression contains derivatives of the Faraday tensor which cannot be replaced by means of the Maxwell equations. These derivatives enter into the principal part of the evolution equations and destroy the hyperbolicity of the evolution equations for the magnetic part of the Weyl tensor. In order to deal with this difficulty, in [8] an additional variable, corresponding to the derivative of the Faraday tensor has been introduced. This will be discussed in subsection 4.5.

In order to obtain the contribution of the Klein-Gordon field to the evolution equation of  $\hat{B}_{ab}$ , we use equation (10) in equation (72). The last two terms of equation (72) with  $T_{ab}$  replaced by  $T_{ab}^{KG}$  are a combination of the fields  $\sigma_a$ ,  $\phi$ ,  $A_a$  and the derivative  $\nabla_a \phi$ . Accordingly, we introduce the following field variable:

$$\varphi_a \equiv \nabla_a \phi. \quad (74)$$

However, the term

$$\frac{\kappa}{2} \mathcal{D}^c (T_{up}^{(KG)} \epsilon^p{}_{cv} (h_a)^u N^v), \quad (75)$$

in equation (72), contains first and second derivatives of the Klein-Gordon field. In order to recast the term (75) in a more convenient form, we first

evaluate  $\nabla_c T_{ab}^{(\text{KG})}$ . Using equation (10) one can write

$$\nabla_c T_{ab}^{(\text{KG})} = \nabla_c T_{ab}^{(\phi)} + \frac{\phi^2}{2} \nabla_c (T_{ab}^{(\sigma)} + T_{ab}^{(\text{A})} + T_{ab}^{(\sigma\text{A})}) + \phi \nabla_c \phi (T_{ab}^{(\sigma)} + T_{ab}^{(\text{A})} + T_{ab}^{(\sigma\text{A})}). \quad (76)$$

Important for our purposes is that the last term in the last equation, namely  $\phi \nabla_c \phi (T_{ab}^{(\sigma)} + T_{ab}^{(\text{A})} + T_{ab}^{(\sigma\text{A})})$ , contains only  $\phi$  and  $\varphi_a$ . Now, consider a *generic* energy momentum tensor

$$T_{ab}^{(\text{X})} = c_{(\text{X})} \left( X_{(a} Y_{b)} - \frac{1}{2} g_{ab} g^{ef} X_e Y_f \right),$$

where  $c_{(\text{X})}$  is a constant. A computation then shows that

$$\begin{aligned} \nabla_c T_{ab}^{(\text{X})} = c_{(\text{X})} & \left( \frac{1}{2} (\nabla_c X_a Y_b + X_a \nabla_c Y_b + \nabla_c X_b Y_a + X_b \nabla_c Y_a) \right. \\ & \left. - \frac{1}{2} g_{ab} g^{ef} \nabla_c X_e Y_f - \frac{1}{2} g_{ab} g^{ef} X_e \nabla_c Y_f \right). \end{aligned} \quad (77)$$

Applying this last formula to the energy-momentum tensors in (11)-(13) one finds that

$$\nabla_c T_{ab}^{(\sigma)} = c_{(\sigma)} \left( (\sigma_a \nabla_c \sigma_b + \sigma_b \nabla_c \sigma_a) - g_{ab} g^{ef} \sigma_e \nabla_c \sigma_f \right), \quad (78)$$

$$\nabla_c T_{ab}^{(\text{A})} = c_{(\text{A})} \left( (A_a \nabla_c A_b + A_b \nabla_c A_a) - g_{ab} g^{ef} A_e \nabla_c A_f \right), \quad (79)$$

$$\nabla_c T_{ab}^{(\phi)} = c_{(\phi)} \left( (\phi_a \nabla_c \phi_b + \phi_b \nabla_c \phi_a) - g_{ab} g^{ef} \phi_e \nabla_c \phi_f \right), \quad (80)$$

and, moreover, that

$$\begin{aligned} \nabla_c T_{ab}^{(\sigma\text{A})} = c_{(\sigma\text{A})} & \left( \frac{1}{2} (A_b \nabla_c \sigma_a + A_a \nabla_c \sigma_b + \sigma_a \nabla_c A_b + \sigma_b \nabla_c A_a) \right. \\ & \left. - \frac{1}{2} g_{ab} g^{ef} (A_f \nabla_c \sigma_e + \sigma_e \nabla_c A_f) \right). \end{aligned} \quad (81)$$

In order to write this last expression in a form suitable to our purposes we introduce the following two auxiliary fields:

$$\psi_{ab} \equiv \nabla_a \sigma_b, \quad \zeta_{ab} \equiv \nabla_a A_b. \quad (82)$$

By introducing these new variables we remove all explicit derivatives from the term (75). Of course, the price paid by this is that we have to find suitable evolution equations for the new auxiliary fields.

#### 4.4 Evolution equation for $\hat{E}_{ab}$

The evolution equation for the electric part of the Weyl tensor can be obtained by expanding the components  $F'_{(a|N|b)} - \frac{1}{2}h_{ab}h^{uv}F'_{uNv}$  of the Friedrich tensor. A lengthy computation shows that

$$\begin{aligned} 0 &= F'_{(a|N|b)} - \frac{1}{2}h_{ab}h^{uv}F'_{uNv} \\ &= \mathcal{L}_N \hat{E}_{ab} + D_c \hat{B}_{d(a\epsilon b)}^{cd} - 2a_c \epsilon^{cd} (\hat{B}_b)_d - 3\chi_{(a}^c \hat{E}_{b)c} - 2\chi_{(a}^c \hat{E}_{b)c} \\ &\quad + h_{ab}\chi^{cd}\hat{E}_{cd} + 2\chi\hat{E}_{ab} + \left( F'_{(a|N|b)}^{(M)} - \frac{1}{2}h_{ab}h^{uv}F'_{uNv}^{(M)} \right), \end{aligned} \quad (83)$$

where the matter contribution is given by

$$\begin{aligned} F'_{(a|N|b)}^{(M)} - \frac{1}{2}h_{ab}h^{uv}F'_{uNv}^{(M)} &= \frac{1}{4}\kappa\mathcal{L}_N \left( T_{dc}h_a^d h_b^c \right) + \frac{1}{2}\kappa \left( h_{(a}^v \mathcal{D}_{b)} - \frac{1}{2}h_{ab}\mathcal{D}^v \right) (T_{uv}N^u) \\ &\quad + \kappa T_{uv}N^v \left( \frac{1}{2}h_{ab}a^u - h_{(b}^u a_{a)} \right) + \frac{1}{2}\kappa T_{uv}N^u N^v \left( \chi_{(ab)} - \frac{h_{ab}}{2}\chi \right) \\ &\quad + \frac{1}{2}\kappa \left( 2\chi_{(a}^u h_{b)}^v - \chi_{(a}^u h_{b)}^v - \frac{1}{2}\chi^{uv}h_{ab} \right) T_{uv}. \end{aligned} \quad (84)$$

In the case of the electromagnetic field the above expression yields explicitly that

$$\begin{aligned} F'_{(a|N|b)}^{(em)} - \frac{h_{ab}h^{uv}}{2}F'_{uNv}^{(em)} &= -\frac{1}{4}\kappa\mathcal{L}_N \left( h_{(a}^q h_{b)}^c F_{qf} F_c^f - \frac{1}{2}h_{ab}F_{qp}F^{qp} \right) - \frac{1}{2}\kappa \left( \frac{h_{ab}\mathcal{D}^v}{2} - h_{(a}^v \mathcal{D}_{b)} \right) E^c F_{vc} \\ &\quad - \kappa E^c F_{vc} \left( h_{(a}^v a_{b)} - \frac{1}{2}h_{ab}a^v \right) - \frac{1}{2}\kappa F_{ud}F_v^d \left( 2\chi_{(a}^v h_{b)}^u - \chi_{(a}^v h_{b)}^u - \frac{1}{2}\chi^{uv}h_{ab} \right) \\ &\quad + \frac{1}{4}\kappa\chi_{(ab)}F_{qp}F^{qp} - \frac{1}{2}\kappa E^2\chi_{(ab)} - \frac{1}{8}\chi h_{ab}F_{qp}F^{qp} + \frac{1}{4}\kappa\chi h_{ab}E^2. \end{aligned} \quad (85)$$

As in the previous subsection we observe the presence of derivatives of the Faraday tensor which need to be dealt with by the introduction of a new field if one is to preserve the hyperbolicity of the equations.

Finally, using equation (10) we obtain the expression for the scalar field contribution. In this case the first two terms of the general expression (84), namely,

$$\frac{1}{4}\kappa\mathcal{L}_N \left( T_{dc}^{(KG)} h_a^d h_b^c \right) + \frac{1}{2} \left( h_{(a}^v \mathcal{D}_{b)} - \frac{1}{2}h_{ab}\mathcal{D}^v \right) T_{uv}^{(KG)} N^u, \quad (86)$$

clearly contain derivatives of the Klein-Gordon energy-momentum tensor. These terms can be rewritten using equation (76) and introducing the auxiliaries fields of equations (74) and (82).

#### 4.5 Evolution equations for the auxiliary field $\psi_{abc}$

The analysis of the evolution equations for the electric and magnetic parts of the Weyl tensor led us to introduce the covariant derivative of the Faraday tensor as further field variable. Accordingly, we set

$$\psi_{abc} \equiv \nabla_a F_{bc}. \quad (87)$$

It has been shown in [8] that, applying a covariant derivative to the Maxwell equations (61), commuting covariant derivatives and using the definition (87) one obtains the following equations for the tensor  $\psi_{abc}$

$$\nabla^b \psi_{adb} = 2F^{eb} R_{[d|ae|b]} - \nabla_a J_d, \quad \nabla^a {}^* \psi_{cab} = \epsilon_b{}^{aud} F_{eu} R_{dac}, \quad (88)$$

where

$${}^* \psi_{abc} \equiv \frac{1}{2} \epsilon_{bc}{}^{ef} \psi_{aef}. \quad (89)$$

As  $\psi_{abc} = \psi_{a[bc]}$ , one can naturally define its *electric* and *magnetic parts* respect to  $N_a$  as

$$\mathcal{E}_{ad} \equiv \psi_{adn} N^n, \quad \mathcal{B}_{au} \equiv \frac{1}{2} \epsilon_u{}^{vzt} N_v \psi_{azt}. \quad (90)$$

Notice that by construction one readily has that  $\mathcal{E}_a N^a = \mathcal{B}_a N^a = 0$ . Projecting the equations in (88) along the directions longitudinal and transverse to  $N^b$  one obtains the following set of evolution equations:

$$\dot{\mathcal{E}}_{e\langle f \rangle} = -2\mathcal{E}_e{}^a h_{f[a} \nabla_{b]} N^b - \epsilon_{abcd} h_f{}^b \nabla^a (\mathcal{B}_e{}^c N^d) + h_f{}^d \mathcal{S}_{ed}, \quad (91)$$

$$\dot{\mathcal{B}}_{e\langle f \rangle} = -2\mathcal{B}_e{}^a h_{f[a} \nabla_{b]} N^b + \epsilon_{abcd} h_f{}^b \nabla^a (\mathcal{E}_e{}^c N^d) - h_f{}^d \mathcal{V}_{ed}. \quad (92)$$

It is noticed that the corresponding constraint equations assume the form

$$D^a \mathcal{B}_{ae} = -\epsilon_{abcd} N^b \mathcal{E}_e{}^c \nabla^a N^d + N^d \mathcal{V}_{ed}, \quad (93)$$

$$D^a \mathcal{E}_{ea} = \epsilon_{abcd} N^a \mathcal{B}_e{}^b \nabla^c N^d - N^d \mathcal{S}_{ed}, \quad (94)$$

where for ease of presentation we have set

$$\mathcal{S}_{fd} \equiv 2F^{eb} R_{[d|fe|b]} - \nabla_f J_d, \quad \mathcal{V}_{fb} \equiv -\epsilon_f{}^{aud} F_{eu} R_{dab}. \quad (95)$$

#### 4.6 Remarks concerning the unknowns and gauge conditions

For convenience, we collect the unknowns discussed in the previous sections in the following vector variable:

$$\mathbf{v} = \{e_a{}^\mu, \Gamma_b{}^a{}_c, \hat{E}_{ab}, \hat{B}_{ab}, A_a, \Phi, E_a, B_a, \mathcal{E}_a, \mathcal{B}_a\}. \quad (96)$$

As already discussed, the matter model under consideration makes further use of field  $A_a$  and the phase of  $\Phi = \phi e^{i\theta}$ . Accordingly one has to introduce the following auxiliary fields:

$$\sigma_a \equiv \nabla_a \theta, \quad \psi_{ab} \equiv \nabla_a \sigma_b, \quad \zeta_{ab} \equiv \nabla_a A_b, \quad (97)$$

Recall also, that in equation (74) we have also introduced the 1-form  $\varphi_a \equiv \nabla_a \phi$ . Following the discussion in [9] we use the real scalar field  $\phi$  to construct a frame adapted to the problem. Accordingly, we assume now that  $\varphi_a$  is timelike and consider a frame satisfying

$$\varphi_a = \alpha e_0, \quad |\alpha| \equiv |\varphi|. \quad (98)$$

In fact, in order to construct the adapted frame for this problem, the gauge choice has been based on some specific regularity assumptions on the scalar field, requiring the timelike vector of the orthonormal frame to follow the matter flow. Firstly, it is assumed that  $\phi \in \mathcal{C}^\infty(\mathcal{M})$  and, secondly we assume  $\nabla_a \phi$  is and remains timelike. In fact, if the gradient is null or spacelike the system evolution breaks down. Hence, one can write  $\varphi^a = \alpha \delta_0^a$ . It follows then that

$$D_i \phi = 0, \quad e'^{\mu}_i \nabla_\mu \phi = 0. \quad (99)$$

In terms of components respect to a coordinate basis one finds that

$$e_0^\mu = \frac{\nabla^\mu \phi}{\alpha}. \quad (100)$$

#### 4.7 Evolution equation for the fields $\phi$ and $\varphi_a$

The evolution equation for the real scalar field  $\phi$  can be recovered from the definition of  $\varphi_a$ . Namely, one has that

$$\partial_t \phi = \varphi_t. \quad (101)$$

An evolution equation for  $\varphi_a$  can be found by using equation (16). One has then that

$$\nabla^a \varphi_a = \phi(\sigma_a \sigma^a + A^2 + 2A^a \sigma_a). \quad (102)$$

In view of the gauge condition in equation (98), equation (102) is, in fact, an equation for  $\alpha$  and the connection coefficients of the frame. The corresponding equations will be discussed in details in Section (4.11).

#### 4.8 The evolution equation for the vector potential $A_a$

The evolution equation for the vector potential  $A_a$  can be readily obtained from the expression of the Faraday tensor in terms of the curl of the vector potential —cf. equation (6). This equation is almost in hyperbolic form. In order to close the system one has to introduce a *gauge source function* —see e.g. [5]. This feature is closely related to the gauge freedom built into equation (6) and allows to specify freely the divergence of  $A_a$ . One has that

$$\nabla_0 A_b = F_{0b} + \nabla_b A_0, \quad (103)$$

$$\nabla^0 A_0 = \zeta(x) - \nabla^i A_i, \quad (104)$$

where  $\zeta(x) \equiv \nabla_a A^a$ . The evolution equations for  $F_{ab}$  have already been discussed in Section 4.2.

#### 4.9 The evolution equation for $\sigma_a$

A key observation for our purposes is that  $\theta$ , the phase of the complex scalar field  $\Phi$ , does not appear explicitly in the evolution equations hitherto discussed, appearing only through its derivative. These derivatives are, in turn, replaced by  $\sigma_a \equiv \nabla_a \theta$ . The evolution equation for the vector potential  $\sigma_a$  can be inferred from the following integrability condition

$$\nabla_a \sigma_b - \nabla_b \sigma_a = 0, \quad (105)$$

as the connection being considered is torsion-free. The structure of this equation is similar to that of equation (6). In order to bring it to hyperbolic form it has to be complemented with information about the divergence of  $\sigma_a$  prescribed by a gauge source function. From equation (105) it follows that

$$\nabla_0 \sigma_b = \nabla_b \sigma_0, \quad (106)$$

$$\nabla^0 \sigma_0 = \psi(x) - \nabla^i \sigma_i, \quad (107)$$

where  $\psi \equiv \nabla^a \sigma_a$ . The gauge source function  $\psi$  is not truly independent. In fact, as a consequence of the Klein Gordon equation (17) one finds that

$$\psi + \zeta = -\frac{2\alpha}{\phi} (\sigma^0 + A^0) e_0. \quad (108)$$

Thus, one readily sees that  $\psi$  can be expressed in terms of  $\zeta$  and further field variables.

#### 4.10 The evolution equations for the auxiliary field $\psi_{ab}$ and $\zeta_{ab}$

In order to motivate the construction of suitable evolution equations for the auxiliary fields  $\psi_{ab}$  and  $\zeta_{ab}$ , we consider first the commutator of covariant derivatives acting on a generic tensor  $X_a$ . One has that

$$\nabla_c \nabla_d X_b - \nabla_d \nabla_c X_b = R^a_{bcd} X_a. \quad (109)$$

Making use of the definition of  $Y_{bc} \equiv \nabla_b X_c$  we rewrite equation (109) as

$$\nabla_c Y_{db} - \nabla_d Y_{cb} = R^a_{bcd} X_a, \quad (110)$$

for the tensor  $Y_{bc}$ . making use of this equation for  $X_a = \sigma_a$ ,  $A_a$  one obtains

$$\nabla_c \psi_{db} - \nabla_d \psi_{cb} = R^a_{bcd} \sigma_a, \quad (111)$$

$$\nabla_c \zeta_{db} - \nabla_d \zeta_{cb} = R^a_{bcd} A_a, \quad (112)$$

were the components of the tensor  $R^a_{bcd}$  in the right-hand side of equations (111) and (112) can be reexpressed, using equation (27), as an algebraic expression involving the Weyl tensor and the matter fields —cf. equations (41)-(42). Thus, the right-hand side is a function of  $\mathbf{v}$  only. The antisymmetry with respect to the covariant derivatives in the left-hand side of equations (111) and (112) suggests that they imply symmetric hyperbolic evolution equations if information concerning the divergence of fields  $\psi_{ab}$  and  $\zeta_{ab}$  is provided. This is done in the next sections.

#### 4.10.1 Evolution equation for the field $\zeta_{ab}$

For convenience, define the tensor  $g_a \equiv \nabla^c \zeta_{ca}$ . If  $g_a$  is explicitly known in terms of other field variables, then suitable hyperbolic equations for  $\zeta_{ab}$  are given by

$$\nabla_0 \zeta_{ia} - \nabla_i \zeta_{0a} = R^e{}_{0ia} A_e, \quad (113)$$

$$\nabla^0 \zeta_{0a} + \nabla^i \zeta_{ia} = g_a(x), \quad (114)$$

The required information about  $g_a$  can be deduced from the Maxwell equation

$$\nabla^a F_{ab} = \nabla^a \nabla_a A_b - \nabla^a \nabla_b A_a = -\phi^2 (\sigma_b + A_b). \quad (115)$$

Commuting covariant derivatives one obtains from the latter that or

$$\nabla^a \zeta_{ab} - \nabla_b \nabla^a A_a - R^{ca}{}_{ab} A_c = -\phi^2 (\sigma_b + A_b). \quad (116)$$

Finally, noticing that  $\zeta(x) = \nabla^a A_a$  and  $\nabla^a \nabla_a A_b = \nabla^a \zeta_{ab} = g_b$  one obtains

$$g_b = \nabla_b \zeta + R^{ca}{}_{ab} A_c - \phi^2 (\sigma_b + A_b).$$

Thus, the divergence  $\nabla^c \zeta_{ca}$  can be rewritten in terms of the derivatives of the gauge source function (which are explicitly known) and an algebraic expression of matter fields.

#### 4.10.2 Evolution equation for the field $\psi_{ab}$

The analysis of the equation (111) is similar to that of equation (112). Assuming that the divergence  $h_a \equiv \nabla^c \psi_{ac}$  is known, one readily obtains the symmetric hyperbolic system

$$\nabla_0 \psi_{da} - \nabla_d \psi_{0a} = R^e{}_{a0d} \sigma_e, \quad (117)$$

$$\nabla_0 \psi_{0a} + \nabla_i \psi_{ia} = h_a. \quad (118)$$

Now, exploiting that  $\psi_{ab} = \psi_{(ab)}$ , one has that

$$\nabla^a \psi_{ab} = \nabla^a \psi_{ba} = \nabla^a \nabla_b \sigma_a = \nabla_b \nabla^a \sigma_a + R^{ea}{}_{ab} \sigma_e.$$

Hence, one finds that

$$h_b = \nabla_b \psi + R^{ea}{}_{ab} \sigma_e.$$

The derivative  $\nabla_b \psi$  can be computed, in turn, using equation (108). Hence, one finds that  $h_b$  can be expressed in terms of the derivatives of the gauge function  $\zeta$  and auxiliary fields.



## 4.11 Evolution equations for frame coefficients and connection coefficients

In order to construct evolution equations for the tetrad and connection coefficients we follow closely the procedure indicated in [9].

First of all, consider equation (24) with  $N = e_0 = \alpha\varphi$  and  $\varphi^\mu = \alpha e_0^\mu$ ,  $\varphi_\mu = e_\mu^0/\alpha$ . In other words, we require the timelike vector of the orthonormal frame to follow the “matter flow lines  $\nabla\phi$ ”, —see Section 4.10 and Section 4.6. Moreover, we introduce a coordinate system  $(x^\mu)$  by requiring the *Lagrange condition*  $N^\mu = e_0^\mu = \delta_0^\mu$  —see Section 4.1). Taking into account the latter and using equation (22) we find that

$$\partial_t e_i^\mu = (\Gamma_0^j{}^i - \Gamma_i^j{}^0) e_j^\mu + \Gamma_0^0{}^i e_0^\mu. \quad (119)$$

The latter is the required evolution equation for  $e_i^\mu$ .

We turn now the attention to the evolution equation for the connection coefficients  $\Gamma_a{}^c{}_b$ . The requirement that the frame components  $e_i^\mu$  are Fermi propagated along  $e_0$  readily implies that  $\Gamma_0^j{}^i = 0$  —see e.g. [6]. Thus, no evolution equations are required for these components. The only components requiring evolution equations are  $\Gamma_q{}^i{}_j$ ,  $\Gamma_0^0{}^i$  and  $\Gamma_i^0{}^j$ .

The evolution equation for  $\Gamma_q{}^i{}_j$  is found using equation (26), where

$$\partial_t \Gamma_q{}^i{}_j = R^i{}_{j0q} - \Gamma_k{}^i{}_j \Gamma_q{}^k{}_0 - \Gamma_0^i{}^j \Gamma_q{}^0{}_j + \Gamma_0^0{}^j \Gamma_q{}^i{}_0. \quad (120)$$

Now, using equation (43) one can express the component of the Riemann tensor  $R^i{}_{j0q}$  in terms of the matter fields. Using the Einstein equation  $R_{0p} = \kappa T_{0p}$ , with  $T_{0p} = T_{0p}^{(\text{em})} + T_{0p}^{(\text{KG})}$  and

$$T_{0p}^{(\text{em})} \equiv -g^{ab} F_{0a} F_{bp}, \quad T_{0p}^{(\text{KG})} = T_{0p}^{(\phi)} + \frac{\phi^2}{2} T_{0p}^{(\sigma)} + \frac{\phi^2}{2} T_{0p}^{(\text{A})} + \frac{\phi^2}{2} T_{0p}^{(\sigma\text{A})}, \quad (121)$$

where

$$T_{0p}^{(\phi)} = \partial_0 \phi \partial_p \phi = 0, \quad T_{0p}^{(\sigma)} = 2\sigma_0 \sigma_p, \quad T_{0p}^{(\text{A})} = 2A_0 A_p, \quad T_{0p}^{(\sigma\text{A})} = 4\sigma_{(0} A_{p)} \quad (122)$$

we have

$$R_{0p} = \kappa \left( -g^{ab} F_{0a} F_{bp} + \phi^2 (\sigma_0 \sigma_p + A_0 A_p + 2\sigma_{(0} A_{p)}) \right). \quad (123)$$

Accordingly, the component  $R_{0p}$  can be expressed purely in terms of undifferentiated field variables.

The evolution equations for the coefficients  $\Gamma_0^0{}^i$  and  $\Gamma_q^0{}^j$  can be found again using equation (26). One finds that

$$\partial_t \Gamma_q^0{}^j = e_q \Gamma_0^0{}^j + R^0{}_{j0q} + \Gamma_0^0{}^j \Gamma_0^0{}^q - \Gamma_i^0{}^j \Gamma_q^i{}^0 - \Gamma_0^0{}^i \Gamma_q^i{}^j. \quad (124)$$

Now, using the conservation equations and the condition  $\nabla_{[a} \nabla_{b]} \phi = 0$  one has the symmetry  $\Gamma_i^0{}^j = \Gamma_j^0{}^i$ . From the Klein Gordon equation for the field  $\phi$  we can recover the following evolution equation for the variable  $\alpha$ :

$$\dot{\alpha} - \alpha \chi - \phi(\sigma^2 + A^2 + 2\sigma^a A_a) = 0. \quad (125)$$

Now, the evolution equation for  $\Gamma_0^0{}_i$  can be found observing that

$$[e_0, e_i]\alpha = \Gamma_0^0{}_i \partial_t \alpha - \chi_i^j \nabla_j \alpha. \quad (126)$$

We can use equation (125) for the term  $\partial_t \alpha$ . From the commutator  $[e_i, e_0]\phi = 0$  we obtain the constraint

$$\nabla_i \alpha - \Gamma_0^0{}_i \alpha = 0. \quad (127)$$

Therefore equation (126) can be rewritten as

$$[e_0, e_i]\alpha = \Gamma_0^0{}_i \alpha \chi - \chi_i^j (\Gamma_0^0{}_j \alpha) + \Gamma_0^0{}_i \phi (\sigma^2 + A^2 + 2q\sigma^a A_a), \quad (128)$$

so as to obtain

$$-(\nabla_0 \Gamma_0^0{}_i + \nabla_i \chi) = \Gamma_0^0{}_i \chi - \chi_i^j \Gamma_0^0{}_j + \frac{2}{\alpha} \Gamma_0^0{}_i \frac{d\bar{\nu}}{d\phi} + \frac{2}{\alpha} \nabla_i \frac{d\bar{\nu}}{d\phi}, \quad (129)$$

where

$$\frac{d\bar{\nu}}{d\phi} \equiv \phi(\sigma^2 + A^2 + 2\sigma^a A_a). \quad (130)$$

The last term of equation (129) contains  $\nabla_i \sigma^a$  and  $\nabla_i A_a$ . However, making use of the auxiliary variables  $\psi_{ab}$  and  $\zeta_{ab}$ , it can be expressed in terms of undifferentiated quantities. Thus, noting that the component  $R_{0p}$  of the Ricci tensor is

$$R_{0p} = -[e_p \Gamma_i^i{}_0 - e_i \Gamma_p^i{}_0 + \Gamma_j^i{}_0 (\Gamma_p^j{}_i - \Gamma_i^j{}_p) + \Gamma_j^i{}_0 \Gamma_i^j{}_p - \Gamma_i^i{}_j \Gamma_p^j{}_0], \quad (131)$$

and using equation (131) in equation (123) it follows

$$- [e_p \Gamma_i^i{}_0 - e_i \Gamma_p^i{}_0 + \Gamma_j^i{}_0 (\Gamma_p^j{}_i - \Gamma_i^j{}_p) + \Gamma_j^i{}_0 \Gamma_i^j{}_p - \Gamma_i^i{}_j \Gamma_p^j{}_0] = \kappa [-g^{ab} F_{0a} F_{bp} + \phi^2 (\sigma_0 \sigma_p + A_0 A_p + 2\sigma_{(0} A_{p)})]. \quad (132)$$

Substituting in equation (129), the term  $(-\nabla_i \chi)$ , where  $\chi = -h^{cd} \Gamma_{cd}^0$ , and noting that  $-e_p \Gamma_i^i{}_0 = e_p h^{ij} \Gamma_{ij}^0$  we have, from equation (132)

$$\nabla_p \chi = -e_i \Gamma_p^i{}_0 + \Gamma_j^i{}_0 (\Gamma_p^j{}_i - \Gamma_i^j{}_p) + \Gamma_j^i{}_0 \Gamma_i^j{}_p - \Gamma_i^i{}_j \Gamma_p^j{}_0 + \quad (133)$$

$$\kappa [-g^{ab} F_{0a} F_{bp} + \phi^2 (\sigma_0 \sigma_p + A_0 A_p + 2\sigma_{(0} A_{p)})], \quad (134)$$

thus equation (129) reads

$$\nabla_0 \Gamma_0^0{}_i - e_p \Gamma_i^p{}_0 = -(\Gamma_0^0{}_i \chi - \chi_i^j \Gamma_0^0{}_j + \frac{2}{\alpha} \Gamma_0^0{}_i \frac{d\bar{\nu}}{d\phi} + \frac{2}{\alpha} \nabla_i \frac{d\bar{\nu}}{d\phi}) \quad (135)$$

$$- \left\{ \Gamma_j^p{}_0 \Gamma_p^j{}_i - \Gamma_p^p{}_j \Gamma_i^j{}_0 + \kappa [-g^{ab} F_{0a} F_{bi} + \phi^2 (\sigma_0 \sigma_i + A_0 A_i + 2\sigma_{(0} A_{i)})] \right\},$$

where we used  $\Gamma_{ij}^0 = \Gamma_j^0{}_i$ . Equation (135) is therefore the evolution equation for the connection  $\Gamma_0^0{}_i$ .

We concentrate now our analysis on equation (124) for the case of the components  $\Gamma_q^0{}_j$ . Using equation (27), the curvature component  $R_{j0q}^0$  can be split as

$$R_{j0q}^0 = C^0{}_{j0q} + \frac{1}{2} (S_{qj} + g_{jq} S_0^0). \quad (136)$$

Thus, one finds that

$$\partial_t \Gamma_q^0{}_j - e_q \Gamma_0^0{}_j = C^0{}_{j0q} + \frac{1}{2} (S_{qj} + g_{jq} S_0^0) + \Gamma_0^0{}_j \Gamma_0^0{}_q - \Gamma_i^0{}_j \Gamma_q^i{}_0 - \Gamma_0^0{}_i \Gamma_q^i{}_j, \quad (137)$$

where

$$S_{ab} = S_{ab}^{(\text{KG})} + S_{ab}^{(\text{em})}, \quad S_{ab} \equiv \kappa (T_{ab} - \frac{1}{3} g_{ab} g^{cd} T_{cd}). \quad (138)$$

Using equations (4), (9) and (121)–(122), one can readily conclude that the right hand side of equation (137) does not contain derivatives of the field unknowns.

Summarizing, equations (135) and (137) provide the required evolution equations for the connection components  $\Gamma_0^0{}_i$  and  $\Gamma_q^0{}_j$ , respectively.

#### 4.12 Summary of the analysis

We now summarize the analysis of the evolution equations carried out in the preceding sections. The unknowns of the evolution equations can be conveniently grouped in the vector

$$\mathbf{v} = (e_i{}^\mu, \Gamma_0^0{}_i, \Gamma_q^i{}_j, \Gamma_q^0{}_j, \hat{E}_{ab}, \hat{B}_{ab}, E_a, B_a, \mathcal{E}_a, \mathcal{B}_a, \phi, A_a, \sigma_a, \alpha, \psi_{ab}, \zeta_{ab}). \quad (139)$$

The components of  $\Gamma_a^b{}_c$  not included in this list are determined by means of gauge conditions and symmetries. By construction, the electric and magnetic parts of the Weyl tensor are tracefree. This symmetry is disregarded in these considerations and is recovered by imposing it on the initial data. It can be shown that if these tensors are initially tracefree, then they will be also tracefree for all later times —see e.g. [7].

The evolution equations for the independent components of the unknowns in (139) and the underlying assumptions in their construction are given as follows:

- (i) The evolution equation for the tetrad coefficients,  $e_i{}^\mu$ , is given by equation (119) by virtue of the Lagrange condition  $N^a = e_0^a = \delta_0^a$ . Equation (119) has the same principal part than the corresponding equation in the case of an uncharged perfect fluid —see e.g. [4]. It gives rise to a symmetric hyperbolic subsystem of equations.

- (ii) The evolution equations for the connection coefficients,  $\Gamma_{0i}^0$  and  $\Gamma_{ib}^a$  are given, respectively, by equations (135) and (137). As a consequence of the gauge conditions one has, in addition, that  $\Gamma_{0i}^j = 0$ . Equation (120) also takes care of  $\Gamma_{jk}^q$ . As in the case of the equations frame coefficients, equations (135), (137) and (120) have the same principal part as those of the analysis given in [4]. Again, one has an hyperbolic subsystem of equations.
- (iii) The evolution equations for the electric and magnetic parts of the Weyl tensor,  $\hat{E}_{ab}$  and  $\hat{B}_{ab}$ , are given, respectively, by equations (83) and (71). As mentioned before, the tracefreeness of these tensors is not used to reduce the number of independent components. Thus one has 12 equations for equal number of components. Equations with a principal part of the form of equations (83) and (71) are symmetric hyperbolic independently of the gauge choice —see e.g. [7].
- (iv) The evolution equation for the electric and magnetic parts of the Faraday tensor,  $E_a$  and  $B_b$ , are given, respectively, by equations (67) and (68). As in the case of the equations for the electric and magnetic parts of the Weyl tensor, the principal part of these equations is known to be hyperbolic independently of the gauge —again, see e.g. [7].
- (v) The evolution equations for the electric and magnetic parts of the auxiliary field  $\psi_{abc}$  (encoding the covariant derivative of the Faraday tensor),  $\mathcal{E}_a$  and  $\mathcal{B}_b$  are given, respectively, by equations (91) and (92). These equations involve 24 equations for as many unknowns. Their structure is analogous to that of equations (67) and (68), except that they contain an extra free index. As a consequence, their principal part gives rise to a symmetric hyperbolic subsystem.
- (vi) The evolution equation for the vector potential  $A_a$  is given by equations (103) and (104). The symmetric hyperbolicity of the equations is implemented by the addition of the (arbitrary) gauge source function  $\zeta(x)$  which allows to freely specify the divergence  $\nabla_a A^a$ .
- (vii) The evolution equation for the scalar field  $\phi$ , is given by equation (101).
- (viii) The evolution equations for the auxiliary field  $\sigma_a \equiv \nabla_a \theta$  are given by equations (106) and (107) with the understanding that the derivative  $\nabla_a \sigma^a = \psi(x)$  arising in the right-hand sides is recovered from the Klein Gordon equation in the form given by (108).
- (ix) The equation for the scalar  $\alpha$  defined by the conditions  $\alpha^2 \equiv \varphi$  and  $\varphi^2 = \nabla_a \phi \nabla^a \phi$  is given by equation (125). This evolution equation is a consequence of the Klein Gordon equation for the field  $\phi$  —see equation (16).
- (x) The evolution equations for the components of  $\psi_{ab} \equiv \nabla_a \psi_b$  are given by equations (117) and (118).
- (xi) Finally, the evolution equations for the components of  $\zeta_{ab} \equiv \nabla_a A_b$  are given by equations (113) and (114).

Summarizing, equations. (119), (135), (137), (120), (83), (71), (67), (68), (91), (92), (103), (104), (106), (107), (125), (117), (118), (113) and (114),

provide the desired symmetric hyperbolic evolution system for the Einstein-charged scalar field system. This system can be written in a concise form using matrix notation. One has that

$$\mathbf{A}^0 \partial_0 \mathbf{v} - \mathbf{A}^j \partial_j \mathbf{v} = \mathbf{B} \mathbf{v}, \quad (140)$$

where  $\mathbf{A}^\mu = \mathbf{A}^\mu(x^\mu, \mathbf{v})$  are matrix-valued functions of the coordinates and the unknowns  $\mathbf{v}$ , and  $\mathbf{A}^0$  is positive definite at least close to a fiducial initial hypersurface. The structure of the system (140) ensures the existence of local solutions to the evolution equations. The analysis of whether these solutions give rise to a solution of the full Einstein-charged scalar field system requires the analysis of the evolution of the constraint equations. This is a computationally intensive argument which will be omitted here. A general argument to handle the propagation of the constraints without having to resort to lengthy computations can be found in [1].

## 5 Conclusion

In the present article we have revisited the issue of well-posedness initial value problem for the evolution equations of the Einstein-Maxwell-Klein-Gordon system (a self-gravitating charged scalar field). The approach followed makes use of the well known  $1 + 3$  tetrad formalism by means of which the various tensorial quantities and equations are projected along the direction of the co-moving observer and onto the orthogonal subspace. Following [4, 7], we require the timelike vector of the orthonormal frame to follow the matter flow lines (Lagrangian gauge). The adapted frame for this problem has been constructed taking  $e_0 = \varphi_a / \alpha$ . The gauge condition depends therefore on the scalar field evolution, the construction of the appropriate frame for this problem relies on suitable regularity conditions for  $\phi$ . As discussed in Section 4.6 it is assumed that  $\phi \in \mathcal{C}^\infty(\mathcal{M})$  and  $\nabla_a \phi$  is timelike. Moreover, we assume the vector fields tetrad to be Fermi transported in the direction of  $U$ , these conditions fix certain components of the connection.

A key feature of our analysis was the introduction of several auxiliary fields, the tensors  $\psi_{abc}, \zeta_{ab}, \psi_{ab}$  corresponding to the covariant derivative of the Faraday tensor  $F_{ab}$ , the vector potential  $A_a$  and the scalar field  $\phi$ . The purpose of introducing these tensors was to ensure the symmetric hyperbolicity of the evolution equations for the components of the Weyl tensor.

## Acknowledgments

DP gratefully acknowledges financial support from the A. Della Riccia Foundation and Blanceflor Boncompagni-Ludovisi née Bildt Foundation.

## References

1. O. Reula, *Hyperbolic methods for Einstein's equations*, Living Rev. Rel. **3**, 1 (1998).
2. S. L. Liebling and C. Palenzuela, *Dynamical bosons stars*, Living Rev. Rel. **15** 6 (2012).
3. F. E. Schunck and E. W. Mielke, *Topical Review: General relativistic boson stars* Class. Quant. Grav. **20** (2003) R301.
4. H. Friedrich, *Evolution equations for gravitating ideal fluid bodies in general relativity*, Phys. Rev. D **57**, 2317 (1998).
5. H. Friedrich, *On the global existence and the asymptotic behaviour of solutions to the Einstein-Maxwell-Yang-Mills equations*, J. Diff. geom. **34**, 275 (1991).
6. H. Friedrich, *Hyperbolic reductions for Einstein's equations*, Class. Quantum Grav. **13**, 1451 (1996).
7. H. Friedrich & A. D. Rendall, *The Cauchy problem for the Einstein equations*, Lect. Notes. Phys. **540**, 127 (2000).
8. D. Pugliese and J. A. Valiente Kroon, *On the evolution equations for ideal magneto-hydrodynamics in curved spacetime*, Gen. Rel. Grav **44**, 2785 (2012).
9. A. Alho, F. C. Mena, & J. A. Valiente Kroon, *The Einstein-Friedrich-nonlinear scalar field system and the stability of scalar field Cosmologies*, In [arXiv:1006.3778](https://arxiv.org/abs/1006.3778), 2010.
10. S. W. Hawking, G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, 1975.
11. J. D. Barrow, R. Maartens, & C. G. Tsagas, *Cosmology with inhomogeneous magnetic fields*, Phys. Rep. **449**, 131 (2007).
12. G. F. R. Ellis & H. van Elst, *Cosmological models: Cargese lectures 1998*, NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. **541**, 1 (1998).
13. C. W. Misner, K. S. Thorne, & J. A. Wheeler, *Gravitation*, W. H. Freeman, 1973.